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Note

A note on graphs with large girth and small minus domination number[†]

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Abstract

Dunbar et al. (1998) in Ref. [3] introduced the *minus domination number* $\gamma^-(G)$ of a graph G and two open problems. In this paper, we show that for every negative integer k and positive integer $m \geq 3$, there exists a graph G with girth m and $\gamma^-(G) \leq k$ which is a positive answer for the open problem 2 in Ref. [3]. © 1999 Elsevier Science B.V. All rights reserved.

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Throughout this paper, let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$ is the set $N(v)$ of vertices adjacent to v . The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. We use $|X|$ for the cardinality of a set X . The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *Betti number* of G . The *girth* of a graph G , $\text{gir}(G)$, is the number of edges of a shortest cycle in G .

A graph \tilde{G} is called a *covering* of G with projection $p: \tilde{G} \rightarrow G$ if there is a surjection $p: V(\tilde{G}) \rightarrow V(G)$ such that $p|_{N(\tilde{v})}: N(\tilde{v}) \rightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We also say that the projection $p: \tilde{G} \rightarrow G$ is an *n-fold covering* of G if p is n to one. A covering $p: \tilde{G} \rightarrow G$ is said to be *regular* (simply, *\mathcal{A} -covering*) if there is a subgroup \mathcal{A} of the automorphism group $\text{Aut}(\tilde{G})$ of \tilde{G} acting freely on \tilde{G} so that the graph G is isomorphic to the quotient graph \tilde{G}/\mathcal{A} , say by h , and the quotient map $\tilde{G} \rightarrow \tilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h . The fiber of an edge or a vertex is its preimage under p . Every \mathcal{A} -covering of a graph G can be constructed as follows [4, 5]: Every edge of a graph G gives rise to a pair of edges in opposite

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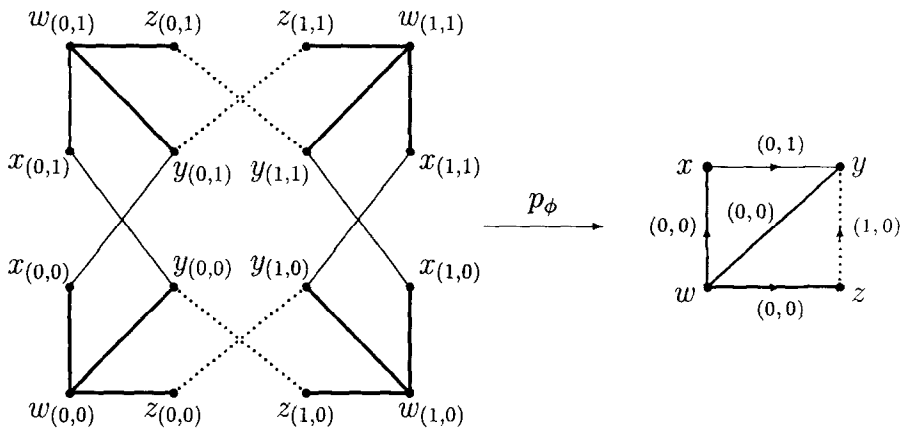


Fig. 1. A $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ -voltage assignment and the derived graph.

directions, i.e., if $\{u, v\}$ is an edge of G , then it gives rise to two directed edges uv and vu . We denote the set of directed edges of G by $D(G)$. An \mathcal{A} -voltage assignment on G is a function $\phi: D(G) \rightarrow \mathcal{A}$ with the property that $\phi(vu) = \phi(uv)^{-1}$ for each $uv \in D(G)$. The values of ϕ are called *voltages*. The *derived graph* $G \times_\phi \mathcal{A}$ derived from an \mathcal{A} -voltage assignment $\phi: D(G) \rightarrow \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$, so that an edge of $G \times_\phi \mathcal{A}$ joins a vertex (u, g) to $(v, g\phi(uv))$ for $uv \in D(G)$ and $g \in \mathcal{A}$. The first coordinate projection $p_\phi: G \times_\phi \mathcal{A} \rightarrow G$, called the *natural projection*, commutes with the right multiplication action of the $\phi(e)$ and the left action of \mathcal{A} on the fibers, which is free and transitive, so that $p_\phi: G \times_\phi \mathcal{A} \rightarrow G$ is an \mathcal{A} -covering. In Fig. 1 a vertex (u, g) of the derived graph $G \times_\phi \mathcal{A}$ is denoted by u_g .

The *minus dominating function* was defined by Dunbar et al. [3] as follows: For any real-valued function $f: V(G) \rightarrow \mathbb{R}$ and $X \subset V(G)$, we define $f(X)$ by the number $\sum_{u \in X} f(u)$. A three-valued function f defined on the vertices of a graph G , $f: V(G) \rightarrow \{-1, 0, 1\} \subset \mathbb{R}$, is a minus dominating function on G if $f(N[v]) \geq 1$ for each $v \in V(G)$. The *minus domination number* for a graph G is

$$\gamma^-(G) = \min\{f(V(G)) \mid f \text{ is a minus dominating function on } G\}.$$

There are several graphs with minus domination numbers which are positive, negative or zero. Zelinka [6] gave a lower bound of a minus domination number for a cubic graph and Dunbar et al. [3] did the same work for regular graphs. In [3], Dunbar et al. studied some properties of the minus domination numbers and attempted to classify graphs according to their minus domination numbers. Moreover, they introduced some open problems. The purpose of this paper is to give a positive answer for one of their open problems.

Main theorem. For every negative integer k and positive integer $m \geq 3$ there exists a connected graph G with $\text{gir}(G) = m$ and $\gamma^-(G) \leq k$.

To prove our main theorem, we start with the following lemma.

Lemma 1. *Let k be a negative integer and m a positive integer greater than 3. Then the following are equivalent.*

- (1) *There exists a connected graph G with $\text{gir}(G) = m$ and $\gamma^-(G) \leq k$.*
- (2) *There exists a connected graph G with $\text{gir}(G) \geq m$ and $\gamma^-(G) \leq k$.*

Proof. The implication (1) \Rightarrow (2) is clear. To prove the implication (2) \Rightarrow (1), let G be such a graph. Since $\gamma^-(G) \leq k$, there exists a minus dominating function f with $f(V(G)) \leq k$. By the definition of minus dominating function f , there exists a vertex v of G such that $f(v) = 1$. Now, we construct a graph H as follows: Let G_1, G_2, \dots, G_m be m copies of G . For each $i = 1, 2, \dots, m$, let v_i be the vertex of G_i corresponding to v . Let H be the graph whose vertex set is $\bigcup_{i=1}^m V(G_i)$ and edge set is $\bigcup_{i=1}^m E(G_i) \cup \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_m, v_1\}\}$, where the unions are disjoint. Then H is the desired graph which completes the proof. \square

Next, we show that for every negative integer k and positive integer $m \geq 3$ there exists a connected graph G with $\text{gir}(G) \geq m$ and $\gamma^-(G) \leq k$. The following lemma compares $\gamma^-(G)$ with $\gamma^-(\tilde{G})$ when \tilde{G} is a covering graph of G .

Lemma 2. *Let n be a natural number and \tilde{G} an n -fold covering graph of G . Then*

$$\gamma^-(\tilde{G}) \leq n\gamma^-(G).$$

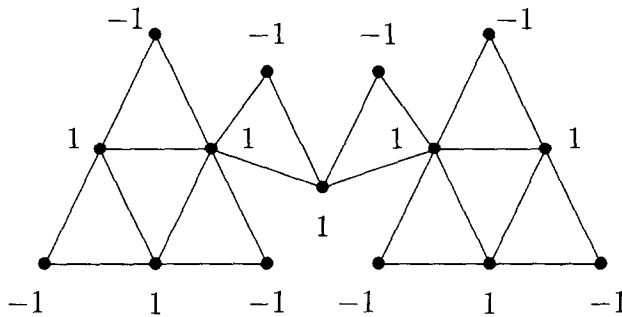
Proof. Let $p: \tilde{G} \rightarrow G$ be the n -fold covering projection and let f be a minus dominating function on G such that $f(V(G)) = \gamma^-(G)$. We define $\tilde{f}: V(\tilde{G}) \rightarrow \{-1, 0, 1\}$ by $\tilde{f}(\tilde{v}) = f(p(\tilde{v}))$ for each $\tilde{v} \in V(\tilde{G})$. Since $p|_{N(\tilde{v})}: N(\tilde{v}) \rightarrow N(p(\tilde{v}))$ is a bijection, \tilde{f} is a minus dominating function on \tilde{G} . Since

$$\tilde{f}(V(\tilde{G})) = \sum_{\tilde{v} \in V(\tilde{G})} \tilde{f}(\tilde{v}) = n \sum_{v \in V(G)} f(v) = nf(V(G)) = n\gamma^-(G),$$

we have $\gamma^-(\tilde{G}) \leq n\gamma^-(G)$. \square

For convenience, let \mathbb{Z}_n be the additive cyclic group of order n and let $l\mathbb{Z}_n$ be the direct sum of l copies of \mathbb{Z}_n . For each $i = 1, 2, \dots, l$, let \mathbf{e}_i be the element in $l\mathbb{Z}_n$ such that the i th entry is 1 and the others are 0. A walk $W = v_0 e_1 v_1 \cdots e_n v_n$ in a graph G is an alternating sequence of vertices and directed edges such that e_i runs from the vertex v_{i-1} to the vertex v_i . A walk $W = v_0 e_1 v_1 \cdots e_n v_n$ in a graph G is *reduced* if for each $i = 1, 2, \dots, n-1$ the directed edge e_{i+1} is not the reverse of the directed edge e_i .

Lemma 3. *Let G be a connected graph which is not a tree. Then there exists a connected graph H such that $\beta(H) = 2^{\beta(G)}(\beta(G) - 1) + 1$, $\text{gir}(H) = 2\text{gir}(G)$ and $\gamma^-(H) \leq 2^{\beta(G)}\gamma^-(G)$.*

Fig. 2. A graph G with $\gamma^-(G) \leq -1$.

Proof. Let T be a spanning tree of G . Then $|E(G) - E(T)| = \beta(G)$. We define a $\beta(G)\mathbb{Z}_2$ -voltage assignment $\phi: D(G) \rightarrow \beta(G)\mathbb{Z}_2$ as follows: For each $\{u, v\}$ in $E(T)$, $\phi(uv) = \phi(vu) = 0$. Let $E(G) - E(T) = \{\{u_1, v_1\}, \dots, \{u_{\beta(G)}, v_{\beta(G)}\}\}$. For each $i = 1, \dots, \beta(G)$, we define $\phi(u_i v_i) = \phi(v_i u_i) = \mathbf{e}_i$. Then $G \times_{\phi} \beta(G)\mathbb{Z}_2$ is the desired graph. Indeed, let (u, g) and (v, h) be a pair of vertices of $G \times_{\phi} \beta(G)\mathbb{Z}_2$. Since the voltages of the edges in $D(G) - D(T)$ generate $\beta(G)\mathbb{Z}_2$, there exists a walk W from u to v in G such that the sum of voltages of the edges in W is $h - g$. By the construction of $G \times_{\phi} \beta(G)\mathbb{Z}_2$, there exists a walk \tilde{W} (a lifting of W) from (u, g) to (v, h) in $G \times_{\phi} \beta(G)\mathbb{Z}_2$ and so $G \times_{\phi} \beta(G)\mathbb{Z}_2$ is connected. Since $|\beta(G)\mathbb{Z}_2| = 2^{\beta(G)}$, $G \times_{\phi} \beta(G)\mathbb{Z}_2$ is a $2^{\beta(G)}$ -fold covering graph of G . Now, by the definition of Betti number of a graph, $\beta(G \times_{\phi} \beta(G)\mathbb{Z}_2) = 2^{\beta(G)}(|E(G)| - |V(G)|) + 1 = 2^{\beta(G)}(\beta(G) - 1) + 1$ and, by Lemma 2, $\gamma^-(G \times_{\phi} \beta(G)\mathbb{Z}_2) \leq 2^{\beta(G)} \gamma^-(G)$. To complete the proof, we need to show that $\text{gir}(G \times_{\phi} \beta(G)\mathbb{Z}_2) = 2\text{gir}(G)$. Let \tilde{C} be a directed cycle in $G \times_{\phi} \beta(G)\mathbb{Z}_2$. Then $p_{\phi}(\tilde{C})$ is a reduced closed walk in G such that the sum of voltages of the edges in $p_{\phi}(\tilde{C})$ is the identity element of $\beta(G)\mathbb{Z}_2$ and so there exists an edge $\{u, v\}$ in $E(G) - E(T)$ such that $|\{i \mid e_i \text{ is an edge in } p_{\phi}(\tilde{C}), e_i = uv \text{ or } vu\}|$ is even. Since $p_{\phi}(\tilde{C})$ is reduced, the length of $p_{\phi}(\tilde{C})$ is greater than or equal to $2\text{gir}(G)$. Note that the length of $p_{\phi}(\tilde{C})$ is equal to that of \tilde{C} . It implies that $\text{gir}(G \times_{\phi} \beta(G)\mathbb{Z}_2) \geq 2\text{gir}(G)$. Let C be a directed cycle in G whose length is $\text{gir}(G)$ and let α denote the sum of voltages of the edges in C . Then, by the definition of ϕ , α is non-identity and is of order two. Let v be a fixed vertex in C . Then the lifting of C beginning at $(v, (0, \dots, 0))$ must terminate at (v, α) and that beginning at (v, α) must terminate at $(v, (0, \dots, 0))$. Since the two liftings are paths of length $\text{gir}(G)$ and disjoint except their origins and terminuses, $G \times_{\phi} \beta(G)\mathbb{Z}_2$ contains a cycle of length $2\text{gir}(G)$. It completes the proof. \square

It is known [3] that there exist many graphs G such that G is not a tree and $\gamma^-(G) < 0$. For example, the graph G in Fig. 2 is one of them.

Note that $\text{gir}(G) = 3$ for the graph G in Fig. 2. Now, by applying Lemma 3 to the graph G in Fig. 2 sufficiently many times, we have the following theorem which is equivalent to our main theorem.

Theorem 1. *For any negative integer k and positive integer $m \geq 3$ there exists a connected graph G with $\text{gir}(G) \geq m$ and $\gamma^-(G) \leq k$.*

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